Complex Analysis **Midterm** 10/16/23

Boundaries This is a take-home exam. It is due in my office by Wednesday, 10/18/23, by 4pm. (You may hand it in earlier than that.) Until then, you have unlimited time to work on it. I expect a legible, correctly-ordered, written-on-one-side-only, stapled hard-copy turned in to me in person.

Legal things to consult Gamelin's book, class notes, returned homework, and questions to me via email or in person. You may also feel free to whine in a general fashion about how hard and unfair the test is to anyone not in the class who is willing to listen to you whine. **Everything else is forbidden** and violations will result in lowered grades and/or J-board hearings. In particular, this means NO consulting other people and NO use of ANYTHING on the internet. Also, if I am convinced that even one person cheated, the final will be 'in-class' during the scheduled three hours. (I don't need proof for this; I just need to believe that at least one person cheated.)

License You are licensed to use any theorem or idea we have done in class, or you have done in your homework, or is in the sections we've covered in Gamelin. This means that if you find some theorem in Gamelin that perfectly answers a question, but it's in, say, Chapter VIII, it may not be used.

Externalities As usual, I will check email as obsessively as possible, and below are extra office hours for the exam. I reserve the right to set a point-price to answer questions. By all means, ask any questions you may have; I'll let you know if it costs to get an answer. Moreover, for some problems, I may be willing to give a hint for points, but you should think of this as a last resort. Clarifications remain a bargain: Completely free! Finally, again, the exam MUST be given to me in person and I will not accept a late exam.

Note Each office hour visit is 5 minutes only. If there are others waiting, you may get back in line. If no one else is waiting, you may come back after 10 minutes have passed.

Office Hours:

Monday	2:30-3:20
Tuesday	1:30-2:20, 3-3:50
Wednesday	12:30-1:20, 1:30-2:20, 3-3:50

Do your best! Show all your work!
Each problem is worth 15 points!
Write your name on the back of your hardcopy!

1. Consider the function $f(z) = z^{1/8}$. Define a suitable domain for f and give a comprehensive explanation of the range of f. What, in totality, is $f(1)$?

2. Consider the set of complex numbers $M = \{x + ix \mid x \in \mathbb{R}\}$. What does the image of M look like under the complex exponential function, e^z? (In a slight abuse of notation, what is $e^{[M]}$?)

3. Explicitly show that the function $f(z) = |z|$ is nowhere differentiable, while by comparison, $g(z) = |z|^2$ is differentiable exactly at 0.

4. (pt. I) Define an appropriate domain for $\log(z)$ such that the set $\mathbb{R}^-$, the negative reals, is in the domain. Explicitly find $\log(-e^2)$.

(pt. II) In $\mathbb{R}^+$, an important "Log Law" is that $\log(ab) = \log(a) + \log(b)$. Using the standard branch of the log, give an explicit example showing the failure of this Log Law in $\mathbb{C}$. Explain clearly why, and under what circumstances, the Log Law fails to be true in $\mathbb{C}$.

5. Suppose $u, v : \mathbb{R}^2 \to \mathbb{R}$ are nice scalar fields, such that $f(z) = f(x+iy) = f(x,y) = u(x,y) + iv(x,y)$ is everywhere analytic. Prove that

$$\overline{f(\overline{z})}\,,$$

the complex conjugate of f evaluated at the complex conjugate of z, is analytic everywhere. Also, using the idea of conformality (the 'amplitwist'), explain as best you can with words and pictures why neither

$$f(\overline{z}) \text{ nor } \overline{\mathrm{f(z)}}$$

are everywhere analytic, but that $\overline{f(\overline{z})}$ is.

6. Show that if $f : \mathbb{C} \to \mathbb{C}$ is analytic, and if the range of f is contained in the unit circle $\{z \in \mathbb{C} \mid |z| = 1\}$, then f must be a constant function.

(Bonus question: 5 points) In fact, let

$$K = \{z_0 + Re^{i\theta} \mid z_0 \in \mathbb{C},\ \theta \in [0, 2\pi]\}$$

be any circle in $\mathbb{C}$. Prove that if the range of f is contained in K, then f must be constant.

7. The *closed upper half plane*, $\mathbb{H} \subset \mathbb{C}$, is defined to be the set of complex numbers $\{a + ib \mid b \geq 0\}$. Find a Fractional Linear Transformation that maps $\mathbb{H}$ exactly onto the closed unit disk $\mathbb{D} = \{z \in \mathbb{C} \mid |z| \leq 1\}$. (FLTs like this turn out to be essential in the modern understanding of hyperbolic geometry.)