Math 365 **Complex Analysis Final** 12/10/23

Boundaries This is a take-home exam. It is due in my office by Wednesday morning, 12/13/23, by 11:30am. (You may hand it in earlier than that.) Until then, you have unlimited time to work on it. I expect a legible, correctly-ordered, written-on-one-side-only, stapled hard-copy turned in to me in person.

Legal things to consult Gamelin's book, class notes, returned homework, and questions to me via email or in person. You may also feel free to whine in a general fashion about how hard and unfair the test is to anyone not in the class who is willing to listen to you whine. **Everything else is forbidden** and violations will result in lowered grades and/or J-board hearings. In particular, this means NO consulting other people and NO use of ANYTHING on the internet.

License You are licensed to use any theorem or idea we have done in class, or you have done in your homework, or is in the sections we've covered in Gamelin. This means that if you find some theorem in Gamelin that perfectly answers a question, but it's in, say, Chapter 8, it may not be used.

Externalities As usual, I will check email as obsessively as possible, and below are extra office hours for the exam. I reserve the right to set a point-price to answer questions. By all means, ask any questions you may have; I'll let you know if it costs to get an answer. Moreover, for some problems, I may be willing to give a hint for points, but you should think of this as a last resort. Clarifications remain a bargain: Completely free! Finally, again, the exam MUST be given to me in person—and I will not accept a late exam.

Note Each office hour visit is 5 minutes only. If there are others waiting, you may get back in line. If no one else is waiting, you may come back after 10 minutes have passed.

Office Hours:

Monday	10:30-11:20, 2-2:50
Tuesday	10:30-11:20, 12:30-1:20, 4:15-4:45
Wednesday	10-10:50, 11-11:20

Do your best! Show as much detail as is appropriate!
Each numbered problem is worth 20 points!
Please write your name on the BACK of the last page of your exam!

1. A function $f(z)$ is doubly-periodic if:

- $\exists\, \omega_0,\ \omega_1 \in \mathbb{C}$, such that for all $c \in \mathbb{R}$, $\omega_0 \neq c * \omega_1$.
- $\forall z \in \mathbb{C}$, it is the case that $f(z + \omega_0) = f(z)$.
- $\forall z \in \mathbb{C}$, it is the case that $f(z + \omega_1) = f(z)$.

Prove that if an entire function is doubly periodic, then it must be constant.

2. Let $f(z)$ and $g(z)$ both be analytic on an open domain that contains a simple loop Γ. Show that if $f = g$ on Γ, then $f = g$ inside of Γ, too.

3. For $z \in [0, 2\pi]$, let $\Gamma = 3i + 2e^{iz}$. Find:

$$\int_\Gamma \sin\left(\frac{1}{z}\right) \cos\left(\frac{1}{z}\right) dz \ .$$

4. For $z \in [0, 2\pi]$, let $\Gamma = 2e^{iz}$. Find:

$$\int_\Gamma \frac{1}{z(z-1)(z-i)(z+1)(z+i)} \ dz \ .$$

5. What is

$$\int_{-\infty}^{\infty} \frac{x^2}{1+x^6} \ dx \quad ?$$

Solve it down to the bitter end. No computers, no calculators.

Bonus Question. (10 points) The function $f(z) = \exp\left(\frac{1}{z}\right)$ has an essential singularity at 0. Picard's Big Theorem says that there must be infinitely many points $\{z_1, z_2, z_3, \dots\}$ inside the unit circle such that $f(z_k) = -1$. Explicitly find infinitely many such points.